A short introduction to one parameter semigroups

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1. INTRODUCTION

According to Lax,[1] semigroups of operators are useful in "partial differential equations describing evolution in time, and flows in dynamical systems". Engel and Nagel[2] state that we develop semigroup theory in light of "autonomous deterministic systems" where the system under consideration is "characterized by a set $\mathcal{Z}$, the state space, of distinct states $z \in \mathcal{Z}$ whose temporal change is to be determined. For example, a planetary system may have the positions and velocities of all planets described as the state space. The *unique motion* of the system is represented by the temporal change of states, i.e. there is a function mapping each $t \in \mathbb{R}$ to a unique state $z(t) \in \mathcal{Z}$." Furthermore we need to describe *autonomous* systems, i.e. the state of the system at time t depends only on some initial state z_0 at some starting time $s < t$ and the elapsed time $\tau = t - s$. The following exposition can be found in the epilogue of Engel and Nagel.[2] Mathematically if we denote the state $z_s(t) = \Phi_{t,s}(z_0)$ where $\Phi : \mathcal{Z} \to \mathcal{Z}$ then the following properties hold

$$\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r} \tag{1.1}$$

$$\Phi_{t,t} = \mathbf{I} \tag{1.2}$$

The transitivity condition (1.1) is due to the system's dependence only on the initial state and the time elapsed, the second condition is trivial, if no time has elapsed the state does not "evolve." If we define

$$\mathbf{T}(t) := \Phi_{r,r-t}$$

Then

$$\begin{cases} \mathbf{T}(0) = \mathbf{I} \\ \\ \mathbf{T}(t)\mathbf{T}(s) = \mathbf{T}(t+s) \end{cases} \tag{1.3}$$

The first condition is obvious. For the second condition

$$\begin{aligned} \mathbf{T}(t)\mathbf{T}(s) &= \Phi_{r,r-t} \circ \Phi_{r,r-s} \\ &= \Phi_{r+t,r} \circ \Phi_{r,r-s} \\ &= \Phi_{r+t,r-s} \quad \text{by the transitivity condition (1.1)} \\ &= \Phi_{r,r-t-s} = \mathbf{T}(t+s) \end{aligned}$$

and these $\mathbf{T}(t)$ define a one-parameter family of mappings on the state space,

$$\{\mathbf{T}(t) : t \in \mathbb{R}\}$$

such a family that satisfies (1.3) when $t \geq 0$ has all the properties of a group except inverses, and it is denoted a one-parameter semigroup. The following is in Lax:[1]

Definition 1. A *one-parameter semigroup* of operators over a complex Banach space X is a family of bounded linear operators $\mathbf{T}(t)$, $t \geq 0$, each mapping $X \to X$, with the following properties:

$$\begin{cases} \mathbf{T}(0) = \mathbf{I} \\ \\ \mathbf{T}(t)\mathbf{T}(s) = \mathbf{T}(t+s) \quad \forall\, t, s \geq 0 \end{cases}$$

2. FURTHER MOTIVATION

$\mathbf{T}(t)$ introduced above can represent many types of operators. Theory needs to be developed to characterize these semigroups. These semigroups can help "define" a solution to the abstract Cauchy problem (ACP):

$$\begin{cases} \frac{d}{dt}x(t) = \mathbf{A}x(t) \\ \\ x(0) = x_0 \end{cases}$$

where $\mathbf{A}$ is some abstract operator. In fact if $\mathbf{A} \in M_n(\mathbb{C})$ is some matrix where $M_n(\mathbb{C})$ is the set of $n \times n$ complex valued matrices then

$$\mathbf{T}(t) = e^{t\mathbf{A}}$$

if and only if $\mathbf{T}$ is differentiable and satisfies

$$\begin{cases} \frac{d}{dt}\mathbf{T}(t) = \mathbf{A}\mathbf{T}(t) \\ \\ \mathbf{T}(0) = \mathbf{I} \end{cases}$$

This is often shown in a graduate level ODE class and will not be proven here. $e^{t\mathbf{A}}$ is defined as follows:

$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{t^k \mathbf{A}^k}{k!} \tag{2.1}$$

(2.1) makes sense here because $\mathbf{A}$ is finite and for each $\mathbf{T} > 0$ such that $|t| \leq \mathbf{T}$

$$\left\| \frac{t^k \mathbf{A}^k}{k!} \right\| \leq \frac{\mathbf{T}^k \|\mathbf{A}\|^k}{k!}$$

so by the Weierstrass M-test (2.1) is absolutely and uniformly convergent. Note that $e^{t\mathbf{A}}$ satisfies the conditions of (1.3) so $\mathbf{T}(t) = e^{t\mathbf{A}}$ forms a one-parameter semigroup. This leads to the questions : What can be said about the ACP when $\mathbf{A}$ is of infinite dimension and possibly unbounded in norm? What is $\mathbf{A}$ under these conditions?

3. SEMIGROUPS WITH RESPECT TO BOUNDED OPERATORS

It can be shown that what is true above for a finite matrix $\mathbf{A}$ is also true if $\mathbf{A} \in B(X)$ is any bounded operator, i.e.

Theorem 3.1. *The following conditions hold :*

(i) Let $\mathbf{A} : X \to X$ *a bounded linear map,* $\mathbf{A} \in B(X)$*. Define* $\mathbf{T}(t)$ *to be*

$$\mathbf{T}(t) = e^{t\mathbf{A}}$$

where the exponential operator is defined as

$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{t^k \mathbf{A}^k}{k!}. \tag{3.1}$$

Then $\mathbf{T}(t)$ *is a one-parameter semigroup of operators, continuous in the operator norm.*

(ii) Conversely let $\mathbf{T}(t) : X \to X$ *be a one-parameter semigroup of operators continuous in the operator norm at* $t = 0$*, i.e.*

$$\lim_{t \downarrow 0} \|\mathbf{T}(t) - \mathbf{I}\| = 0. \tag{3.2}$$

Then $\mathbf{T}(t)$ *is of the form*

$$\mathbf{T}(t) = e^{t\mathbf{A}}$$

where $\mathbf{A}$ *is some bounded linear map from* $X \to X$*.*

Proof. (i) For $t = 0$ have

$$\mathbf{T}(0) = e^{0\mathbf{A}} = \sum_{k=0}^{\infty} \frac{0^k \cdot \mathbf{A}^k}{k!} = \mathbf{I} + \frac{0 \cdot \mathbf{A}}{1!} + \frac{0 \cdot \mathbf{A}^2}{2!} + \cdots = \mathbf{I}.$$

So $\mathbf{T}$ satisfies the first property of definition 1. For $t, s \geq 0$ have

$$\mathbf{T}(t)\mathbf{T}(s) = (e^{t\mathbf{A}})(e^{s\mathbf{A}}) = \left(\sum_{k=0}^{\infty} \frac{t^k \mathbf{A}^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{s^k \mathbf{A}^k}{k!}\right).$$

Since both series converge the Cauchy product (see appendix A) can be used and obtain

$$\begin{aligned}\left(\sum_{k=0}^{\infty} \frac{t^k \mathbf{A}^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{s^k \mathbf{A}^k}{k!}\right) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{t^{n-k}\mathbf{A}^{n-k}}{(n-k)!} \cdot \frac{s^k \mathbf{A}^k}{(k)!} \\ &= \sum_{n=0}^{\infty} \frac{(t+s)^n \mathbf{A}^n}{n!} = \mathbf{T}(t+s)\end{aligned}$$

the last step uses the binomial theorem. So $\mathbf{T}$ satisfies the second property of definition 1. So $\mathbf{T}(t)$ is a one-parameter semigroup, now it must be shown that it is continuous in the operator norm. From what we just proved about the semigroup property, $\forall\, t, h \geq 0$

$$e^{(t+h)\mathbf{A}} - e^{t\mathbf{A}} = e^{t\mathbf{A}}e^{h\mathbf{A}} - e^{t\mathbf{A}} = e^{t\mathbf{A}}(e^{h\mathbf{A}} - \mathbf{I})$$

so it is enough to show that

$$\lim_{h\downarrow 0} e^{h\mathbf{A}} = \mathbf{I}$$

with the limit in the operator norm to show continuity.

$$\begin{aligned}\|e^{h\mathbf{A}} - \mathbf{I}\| &= \left\|\sum_{k=1}^{\infty} \frac{h^k \mathbf{A}^k}{k!}\right\| \\ &\leq \sum_{k=1}^{\infty} \frac{|h|^k \|\mathbf{A}\|^k}{k!} = e^{|h|\|\mathbf{A}\|} - 1\end{aligned}$$

(ii) Some complex analysis will be used for this part. The Taylor expansion of $\log(1+z)$ when $|z+1| < 1$ is

$$\log(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k}$$

so for any $\mathbf{T}$ such that $\|\mathbf{T} - \mathbf{I}\| < 1$ we have

$$\log(\mathbf{T}) = \log(\mathbf{I} + \mathbf{T} - \mathbf{I}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(\mathbf{T} - \mathbf{I})^k}{k} \tag{3.3}$$

The following result will be used: If $\mathbf{T} : X \to X$ and $\mathbf{S} : X \to X$ commute and

$$\|\mathbf{T} - \mathbf{I}\| < \frac{1}{3}$$
$$\|\mathbf{S} - \mathbf{I}\| < \frac{1}{3}$$

then

$$\|\mathbf{TS} - \mathbf{I}\| < 1$$

and

$$\log(\mathbf{TS}) = \log(\mathbf{T}) + \log(\mathbf{S}) \tag{3.4}$$

where the logarithms are defined as above. First

$$\begin{aligned}\|\mathbf{TS} - \mathbf{I}\| &= \|(\mathbf{T} - \mathbf{I})(\mathbf{S} - \mathbf{I}) + (\mathbf{T} - \mathbf{I}) + (\mathbf{S} - \mathbf{I})\| \\ &\leq \|(\mathbf{T} - \mathbf{I})\|\|(\mathbf{S} - \mathbf{I})\| + \|\mathbf{T} - \mathbf{I}\| + \|\mathbf{S} - \mathbf{I}\| \\ &= \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \\ &= \frac{8}{9} < 1\end{aligned}$$

By (3.2) there is $a > 0$ such that $\|\mathbf{T}(t) - \mathbf{I}\| < 1/3$ for $t < a$. Define

$$\mathbf{L}(t) := \log(\mathbf{T}(t)), \quad 0 \leq t < a$$

where the logarithm is as defined in (3.3). From definition 1 $\mathbf{T}(t)$ and $\mathbf{T}(s)$ commute so using (3.4) have

$$\mathbf{L}(t + s) = \mathbf{L}(t) + \mathbf{L}(s), \quad 0 \leq t + s < a$$

this implies that for all $t \in \mathbb{Q}$ such that $t < a$

$$\frac{\mathbf{L}(t)}{t} = \frac{\mathbf{L}(p/q)}{p/q} = \frac{q\mathbf{L}(p/q)}{p} = \frac{\mathbf{L}(p)}{p} = \frac{p\mathbf{L}(1)}{p} = \mathbf{L}(1)$$

i.e. $\mathbf{L}(t)/t$ is independent of t. Denote

$$\mathbf{A} = \frac{\mathbf{L}(t)}{t}, \quad t \in \mathbb{Q},\ 0 \leq t < a$$

then

$$\mathbf{L}(t) = t\mathbf{A}, \quad t \in \mathbb{Q},\ 0 \leq t < a$$

By the second semigroup property

$$\mathbf{T}(t + h) - \mathbf{T}(t) = \mathbf{T}(t)(\mathbf{T}(h) - \mathbf{I})$$

and by assumption $\mathbf{T}(t)$ is continuous at $t = 0$ so the above implies that $\mathbf{T}(t)$ is continuous for all $t \geq 0$. This and (3.3) implies that $\mathbf{L}(t)$ is continuous when $t < a$. Since the rationals are dense, by continuity we have that

$$\mathbf{L}(t) = t\mathbf{A}, \quad t \in \mathbb{R},\ 0 \leq t < a$$

exponentiating yields

$$\mathbf{T}(t) = e^{t\mathbf{A}}, \quad t < a$$

then for any fixed $t > 0$ there is an $n \in \mathbb{N}$ such that $t < na$ then $t/n < a$ and

$$\mathbf{T}(t) = (\mathbf{T}(t/n))^n = (e^{(t/n)\mathbf{A}})^n = e^{t\mathbf{A}}$$

□

4. STRONGLY CONTINUOUS ONE-PARAMETER SEMIGROUPS

We will need the following definitions:

Definition 2. The *resolvent set* of an operator $\mathbf{A}$, denoted $\rho(\mathbf{A})$ is

$$\rho(\mathbf{A}) = \{\lambda : (\lambda - \mathbf{A})^{-1} exists\}$$

Definition 3. The *spectrum* of an operator $\mathbf{A}$, denoted $\sigma(\mathbf{A})$ is

$$\mathbb{C} \backslash \rho(\mathbf{A})$$

Definition 4. The *resolvent* of an operator $\mathbf{A}$, denoted $r_A(\lambda)$ is

$$r_A(\lambda) = (\lambda - \mathbf{A})^{-1}$$

where $\lambda \in \rho(\mathbf{A})$.

Definition 5. A *strongly continuous one-parameter semigroup* of operators over a complex Banach space X is a one-parameter semigroup such that

$$\lim_{t \downarrow 0} \mathbf{T}(t)x = x, \quad \forall\, x \in X \tag{4.1}$$

Theorem 4.1. *If $\mathbf{T}(t)$ is a strongly continuous one-parameter semigroup then*

(i) There exist b, k constants such that $\mathbf{T}(t)$ is bounded in norm

$$\|\mathbf{T}(t)\| \le be^{kt} \tag{4.2}$$

(ii) $\mathbf{T}(t)x$ is a strongly continuous function of t for every $x \in X$.

Proof. (i) First we need that $\|\mathbf{T}(t)\|$ is uniformly bounded in some neighborhood of $t = 0$ (note: Engel and Nagel[2] do not address the boundedness). Suppose that it is not, then there exists a sequence $\{t_j\}$ such that $t_j \to 0$ and

$$\|\mathbf{T}(t_j)\| \to \infty$$

then, by the uniform boundedness principle there must exist some $y \in X$ such that

$$\|\mathbf{T}(t_j)y - y\| \ge c, \ \forall\, j \quad \text{some constant } c$$

which contradicts the strong continuity condition. So for some $b > 0$, some $a > 0$:

$$\|\mathbf{T}(t)\| \le b, \quad t \le a$$

and we can rewrite any t as $t = n \cdot a + r$ where $0 \le r < a$. Then using (1.3) for any t

$$\|\mathbf{T}(t)\| = \|\mathbf{T}(na + r)\| = \|\mathbf{T}^n(a)\mathbf{T}(r)\| \le \|\mathbf{T}(a)\|^n \|\mathbf{T}(r)\| \le b^{n+1}$$

and if we denote $k = \frac{1}{a} \log b$ then $b^{n+1} \le be^{kt}$.

(ii) Fix x. By (1.3)

$$\mathbf{T}(t)x - \mathbf{T}(s)x = \mathbf{T}(s)\big[\mathbf{T}(t-s)x - x\big]$$
$$\Longrightarrow \|\mathbf{T}(t)x - \mathbf{T}(s)x\| \le \|\mathbf{T}(s)\|\|\mathbf{T}(t-s)x - x\| \le be^{ks}\|\mathbf{T}(t-s)x - x\|$$

by the strong continuity condition (4.1) if $|t - s| < \delta$ then

$$be^{ks}\|\mathbf{T}(t-s)x - x\| \le be^{ks}\epsilon$$

so we have strong continuity in t.

□

5. THE INFINITESIMAL GENERATOR OF A ONE-PARAMETER SEMIGROUP

In the strongly continuous case, we need an analogue to the uniformly continuous case for $\mathbf{A} \in B(X)$ where

$$\mathbf{T}(t) = e^{t\mathbf{A}}. \tag{5.1}$$

In the strongly continuous case $\mathbf{A}$ may be unbounded.

Definition 6. Let $\mathbf{T}(t)$ be a strongly continuous one-parameter semigroup of operators mapping $X \to X$. Then the *infinitesimal generator* $\mathbf{A}$ is defined by

$$\mathbf{A}x := \lim_{h\downarrow 0} \frac{\mathbf{T}(h)x - x}{h}. \tag{5.2}$$

and the domain $D(\mathbf{A})$ is defined

$$D(\mathbf{A}) := \{x \in X : \mathbf{A}x \text{ exists}\}. \tag{5.3}$$

Theorem 5.1. *Let $\mathbf{T}(t)$ be a strongly continuous one-parameter semigroup with infinitesimal generator $\mathbf{A}$. Then*

(i) $\mathbf{A}$ commutes with $\mathbf{T}(t)$ that is

$$\mathbf{A}\mathbf{T}(t)x = \mathbf{T}(t)\mathbf{A}x \tag{5.4}$$

and

$$x \in D(\mathbf{A}) \implies \mathbf{T}(t)x \in D(\mathbf{A}). \tag{5.5}$$

(ii) The domain of $\mathbf{A}$ is dense in X.

(iii) The domain of $\mathbf{A}^n$ where $n \in \mathbb{N}$ is dense in X.

(iv) $\mathbf{A}$ is a closed operator.

(v) All complex numbers z where $\mathrm{Re}(z) > k$ belong to the resolvent set of $\mathbf{A}$, $\rho(\mathbf{A})$, where k is the constant in (4.2). The resolvent of $\mathbf{A}$, $(z - \mathbf{A})^{-1}$, is the Laplace transform of $\mathbf{T}$.

Proof. (i) Using the semigroup property (1.3), letting $x \in D(\mathbf{A})$ and using definition 6 we get

$$\begin{aligned} &\frac{\mathbf{T}(t+h) - \mathbf{T}(t)}{h}x = \mathbf{T}(t)\frac{\mathbf{T}(h) - \mathbf{I}}{h}x = \frac{\mathbf{T}(h) - \mathbf{I}}{h}\mathbf{T}(t)x \\ \implies &\lim_{h\downarrow 0} \frac{\mathbf{T}(t+h) - \mathbf{T}(t)}{h}x = \lim_{h\downarrow 0}\mathbf{T}(t)\frac{\mathbf{T}(h) - \mathbf{I}}{h}x = \lim_{h\downarrow 0}\frac{\mathbf{T}(h) - \mathbf{I}}{h}\mathbf{T}(t)x \\ \implies &\frac{d}{dt}\mathbf{T}(t)x = \mathbf{T}(t)\mathbf{A}x = \mathbf{A}\mathbf{T}(t)x \end{aligned} \tag{5.6}$$

(ii) We need an integral equation analogous to (5.6):

$$\mathbf{T}(t)x - x = \mathbf{A}\int_0^t \mathbf{T}(s)x ds \tag{5.7}$$

The integral on the RHS exists because we have strong continuity, therefore $\mathbf{T}(s)x$ is a continuous function of s so the integral can be defined in the Riemannian way. So we just need to prove the equality actually holds. We have that

$$\begin{aligned}
\frac{\mathbf{T}(h)-\mathbf{I}}{h}\int_0^t \mathbf{T}(s)x ds &= \frac{1}{h}\int_0^t [\mathbf{T}(s+h)-\mathbf{T}(s)]x ds \\
&= \frac{1}{h}\int_0^t [\mathbf{T}(s+h)-\mathbf{T}(s)]x ds \\
&= \frac{1}{h}\left(\int_0^t \mathbf{T}(s+h)x ds - \int_0^t \mathbf{T}(s)x ds\right) \\
&= \frac{1}{h}\left(\int_t^{t+h} \mathbf{T}(s)x ds + \int_h^t \mathbf{T}(s)x ds - \int_0^h \mathbf{T}(s)x ds - \int_h^t \mathbf{T}(s)x ds\right) \\
&= \frac{1}{h}\int_t^{t+h} \mathbf{T}(s)x ds - \frac{1}{h}\int_0^h \mathbf{T}(s)x ds
\end{aligned}$$

using strong continuity and letting $h \to 0$ get

$$\mathbf{A}\int_0^t \mathbf{T}(s)x ds = \mathbf{T}(t)x - \mathbf{T}(0)x = \mathbf{T}(t)x - x$$

This implies that for any $x \in X$

$$\int_0^t \mathbf{T}(s)x ds \in D(\mathbf{A})$$

also have that

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^t \mathbf{T}(s)x ds = x \tag{5.8}$$

which implies that $D(\mathbf{A})$ is dense in X

(iii) Omitted see Lax[1] page 422.

(iv) Step 1: We need to show that the following holds for all $x \in X$ (Riemannian integral here)

$$\mathbf{T}(t)x - x = \int_0^t \mathbf{T}(s)\mathbf{A}x ds \tag{5.9}$$

First take the derivative of both sides, the LHS derivative can be taken as in (5.6):

$$\begin{aligned}
\frac{d}{dt}(\mathbf{T}(t)x - x) &= \frac{d}{dt}\int_0^t \mathbf{T}(s)\mathbf{A}x ds \\
\implies \mathbf{T}(t)\mathbf{A}x &= \mathbf{T}(t)\mathbf{A}x
\end{aligned}$$

so the derivatives are equal everywhere. If we plug in $t = 0$ into (5.9) we get

$$\mathbf{T}(0)x - x = \int_0^0 \mathbf{T}(s)\mathbf{A}x ds = 0$$

so at $t = 0$ (5.9) is true and the derivatives are equal everywhere so by the fundamental theorem of calculus (5.9) holds everywhere.

Step 2: Now we need to show that $\mathbf{A}$ is closed. Let $\{x_n\}$ be a sequence such that $x_n \in D(\mathbf{A})$ and $x_n \to x$, $\mathbf{A}x_n \to y$. Then from (5.9) have

$$\mathbf{T}(t)x_n - x_n = \int_0^t \mathbf{T}(s)\mathbf{A}x_n ds$$

the LHS obviously goes to $\mathbf{T}(t)x - x$. By continuity the RHS converges as well. So

$$\mathbf{T}(t)x - x = \int_0^t \mathbf{T}(s)y ds$$

divide by t and let $t \downarrow 0$:

$$\lim_{t\downarrow 0} \frac{\mathbf{T}(t)x - x}{t} = \lim_{t\downarrow 0} \frac{1}{t}\int_0^t \mathbf{T}(s)y ds$$

then from (5.8) we have

$$\lim_{t\downarrow 0} \frac{\mathbf{T}(t)x - x}{t} = y$$

so we have existence of

$$\lim_{t\downarrow 0} \frac{\mathbf{T}(t)x - x}{t}$$

which implies that $x \in D(\mathbf{A})$ and

$$\mathbf{A}x = y.$$

(v) Define the Laplace transform of $\mathbf{T}(t)$ as

$$\mathcal{L}(z)x = \int_0^\infty e^{-zs}\, \mathbf{T}(s)x ds \tag{5.10}$$

where the integral is of the Riemannian type. Since we have the estimate (4.2), $\|\mathbf{T}(s)\| \leq be^{ks}$, then when $\mathrm{Re}(z) > k$ the integral converges and

$$\|\mathcal{L}(z)x\| \leq \int_0^\infty be^{[k-\mathrm{Re}(z)]s}\|x\| ds = \frac{b\|x\|}{\mathrm{Re}(z) - k}$$

so $\mathcal{L}(z)$ is bounded, i.e. from the above equation and the by definition of operator norm

$$\|\mathcal{L}(z)\| \leq \frac{b}{\mathrm{Re}(z) - k} \tag{5.11}$$

We need to show that $\mathcal{L}(z) = r_A(z)$, from which it follows that since the Laplace transform exists for $Re(z) > k$ so does the resolvent of z.

Step 1: Show that $e^{-zt}\mathbf{T}(t)$ is a strongly continuous semigroup and that it has infinitesimal generator $(\mathbf{A} - z\mathbf{I})$ (here $\mathbf{A}$ is still the generator of $\mathbf{T}(t)$).

$$e^{-z\cdot 0}\mathbf{T}(0) = 1 \cdot \mathbf{I} = \mathbf{I}$$
$$e^{-z(t+s)}\mathbf{T}(t+s) = e^{-zt}e^{-zs}\mathbf{T}(t)\mathbf{T}(s) = \left(e^{-zt}\mathbf{T}(t)\right)\left(e^{-zs}\mathbf{T}(s)\right)$$

and

$$\lim_{t\downarrow 0}\frac{e^{-zh}\mathbf{T}(h)x-x}{h} = \lim_{t\downarrow 0}\frac{e^{-zh}\mathbf{T}(h)x-e^{-zh}x-x+e^{-zh}x}{h}$$
$$= \mathbf{A}x - \lim_{t\downarrow 0}\frac{e^{-zh}x-x}{h}$$
$$= \mathbf{A}x - zx$$

Step 2: Now need to show that $\mathcal{L}(z)$ is the inverse of $(z-\mathbf{A})$. Apply (5.7) to $e^{-zt}\mathbf{T}(t)$ and get

$$e^{-zt}\mathbf{T}(t)x - x = (\mathbf{A}-z\mathbf{I})\int_0^t e^{-zs}\mathbf{T}(s)xds.$$

Let $t\to\infty$ in the above equation and have $Re(z)>k$ then from (4.2) the LHS goes to $-x$ and the integral part of the RHS goes to $\mathcal{L}(z)x$ so we get

$$-x = (\mathbf{A}-z\mathbf{I})\mathcal{L}(z)x$$
$$\Longrightarrow x = (z\mathbf{I}-\mathbf{A})\mathcal{L}(z)x$$

Apply (5.9) to $e^{-zt}\mathbf{T}(t)$ and get

$$e^{-zt}\mathbf{T}(t)x - x = \int_0^t e^{-zs}\mathbf{T}(s)(\mathbf{A}-z\mathbf{I})xds = \left(\int_0^t e^{-zs}\mathbf{T}(s)ds\right)(\mathbf{A}-z\mathbf{I})x.$$

Let $t\to\infty$ in the above equation and have $Re(z)>k$ then from (4.2) the LHS goes to $-x$ and the integral part of the RHS goes to $\mathcal{L}(z)x$ so we get

$$-x = \mathcal{L}(z)(\mathbf{A}-z\mathbf{I})x$$
$$\Longrightarrow x = \mathcal{L}(z)(z\mathbf{I}-\mathbf{A})x$$

□

Theorem 5.2. *Let* $\mathbf{T}(t)$ *be a strongly continuous one-parameter semigroup with infinitesimal generator* $\mathbf{A}$. *Then* $\mathbf{T}(t)$ *is uniquely determined by* $\mathbf{A}$.

Proof. See Lax[1] page 424. □

6. THE HILLE-YOSIDA THEOREM

For a strongly continuous one-parameter semigroup that is a contraction, we can reconstruct $\mathbf{T}(t)$ from the infinitesimal generator $\mathbf{A}$. Contraction means

$$\|\mathbf{T}(t)\| \le 1, \quad \forall\, t \ge 0$$

Since the semigroup is uniquely determined by its generator, in the case of the contraction we can completely characterize both the semigroup and its generator. The following theorem answers the question "When is the operator $\mathbf{A}$ the generator of some semigroup $\mathbf{T}(t)$?" The following is mostly in Lax.[1]

Theorem 6.1. *(Hille-Yosida) Let* $\mathbf{T}(t)$ *be a strongly continuous one-parameter semigroup mapping* $X\to X$ *such that*

$$\|\mathbf{T}(t)\| \le 1, \quad \forall\, t \ge 0$$

then

(i) The infinitesimal generator $\mathbf{A}$ *of* $\mathbf{T}(t)$ *has every positive real* λ *in its resolvent set,* $\rho(\mathbf{A})$*, i.e. if* $\lambda \in [0,\infty)$ *then* $\lambda \in \rho(\mathbf{A})$ *and*

$$\|(\lambda - \mathbf{A})^{-1}\| \le \frac{1}{\lambda}. \tag{6.1}$$

(ii) Conversely, if $\mathbf{A}$ *is an unbounded operator, dense in* X *and* $[0,\infty) \subset \rho(\mathbf{A})$ *and*

$$\|(\lambda - \mathbf{A})^{-1}\| \le \frac{1}{\lambda}, \quad \forall\, \lambda \in \rho(\mathbf{A}).$$

Then $\mathbf{A}$ *is the generator of a strongly continuous semigroup* $\mathbf{T}(t)$ *such that*

$$\|\mathbf{T}(t)\| \le 1, \quad \forall\, t \ge 0$$

Proof. (i) From the contraction condition we know that $b = 1$ in theorem (4.1) (i) (see the proof of theorem (4.1) (i)) which implies that $k = \frac{1}{a}log(1) = 0$. From theorem (5.1) (v) we have $r_A(z) = \mathcal{L}(z)$, i.e. the resolvent of z is equal to the Laplace transform of z as defined in (5.10) and from (5.11)

$$\|r_A(z)\| = \|\mathcal{L}(z)\| \le \frac{b}{\mathrm{Re}(z) - k} = \frac{1}{\mathrm{Re}(z)}$$

but since $k = 0$ then by theorem (5.1) (v) have that all z such that $Re(z) > 0$ are in the resolvent set $\rho(\mathbf{A})$ which includes the all positive real numbers λ so

$$\|(\lambda - \mathbf{A})^{-1}\| = \frac{1}{\lambda}, \quad \lambda \in \mathbb{R} \tag{6.2}$$

(ii) This following is due to Yosida, there is a different proof due to Hille. Define the *Yosida approximates*, $\mathbf{A}_n$ as follows

$$\mathbf{A}_n = n\mathbf{A}r_A(n) \tag{6.3}$$

We have the following identity:

$$r_A(n)\mathbf{A} = \mathbf{A}r_A(n) = nr_A(n) - \mathbf{I} \tag{6.4}$$

because

$$\begin{aligned} n(n-\mathbf{A})^{-1} - \mathbf{I} &= (n\mathbf{I} - (n-\mathbf{A}))(n-\mathbf{A})^{-1} = \mathbf{A}(n-\mathbf{A})^{-1}, \quad \text{and} \\ n(n-\mathbf{A})^{-1} - \mathbf{I} &= (n-\mathbf{A})^{-1}(n\mathbf{I} - (n-\mathbf{A})) = (n-\mathbf{A})^{-1}\mathbf{A} \end{aligned}$$

Substituting into (6.3) yields

$$\mathbf{A}_n = n^2 r_A(n) - n \tag{6.5}$$

So for each n by (6.1) we have

$$\|\mathbf{A}_n\| \le n^2\frac{1}{n} + n = 2n$$

i.e. for each n have $\mathbf{A}_n$ is bounded. Define $\mathbf{T}_n(t)$ as

$$\mathbf{T}_n(t) = e^{t\mathbf{A}_n}$$

where the exponential is defined as in (3.1).

Step 1: We need to show when we have (6.1) then for $x \in X$

$$\lim_{n\to\infty} nr_A(n)x = x \tag{6.6}$$

Using (6.4) and (6.1) we get

$$\begin{aligned}\|nr_A(n)x - x\| &= \|(nr_A(n) - \mathbf{I})x\| \\ &= \|r_A(n)\mathbf{A}x\| \\ &\leq \frac{1}{n}\|\mathbf{A}x\| \to 0 \text{ as } n \to \infty\end{aligned}$$

but here we must have $x \in D(\mathbf{A})$ because $\mathbf{A}$ is only densely defined on X. So we have (6.6) for $x \in D(\mathbf{A})$. So have $D(\mathbf{A})$ is dense in X and

$$\|nr_A(n)\| = \|n(n - \mathbf{A})^{-1}\| = n\|(n - \mathbf{A})^{-1}\| \leq \frac{n}{n} = 1$$

by (6.1). So let $y \in X\backslash D(\mathbf{A})$ then from denseness there exists $x \in D(\mathbf{A})$ such that $\|x - y\| < \epsilon/2$ and

$$\begin{aligned}\|nr_A(n)y - y\| &= \|nr_A(n)y - y + nr_A(n)x - x - nr_A(n)x + x\| \\ &\leq \|nr_A(n)y - nr_A(n)x\| + \|x - y\| + \|nr_A(n)x - x\| \\ &\leq \|nr_A(n)\|\|y - x\| + \|x - y\| + \frac{1}{n}\|\mathbf{A}x\| \\ &\leq \epsilon + \frac{1}{n}\|\mathbf{A}x\| \to \epsilon \text{ as } n \to \infty\end{aligned}$$

but ϵ is arbitrary so we have (6.6) for all $x \in X$

Step 2: Next we want to show that for all $x \in D(\mathbf{A})$,

$$\lim_{n\to\infty} \mathbf{A}_n x = \mathbf{A}x \tag{6.7}$$

From (6.4) and (6.3) we have

$$\mathbf{A}_n x = n\mathbf{A}r_A(n)x = nr_A(n)\mathbf{A}x$$

letting $y = \mathbf{A}x \in X$ and using (6.6)

$$\lim_{n\to\infty} \mathbf{A}_n x = \lim_{n\to\infty} nr_A(n)y = y = \mathbf{A}x$$

Step 3: Now we want to show that each $\mathbf{T}_n(t)$ is a contraction. Using (6.5) and (3.1) we get

$$\mathbf{T}_n(t) = e^{t\mathbf{A}_n} = e^{t(n^2 r_A(n) - n)} = e^{-nt}e^{tn^2 r_A(n)} = e^{-nt}\sum_{k=0}^{\infty}\frac{n^{2k}t^k \cdot (r_A(n))^k}{k!}$$

Now use the resolvent condition (6.1) and get

$$\begin{aligned}\|\mathbf{T}_n(t)\| &\leq e^{-nt}\sum_{k=0}^{\infty}\frac{n^{2k}|t|^k \cdot \|r_A(n)\|^k}{k!} \\ &\leq e^{-nt}\sum_{k=0}^{\infty}\frac{n^{2k}|t|^k \cdot}{k!}\frac{1}{n^k} \\ &= e^{-nt}\sum_{k=0}^{\infty}\frac{n^k|t|^k \cdot}{k!} \\ &= e^{-nt}e^{n|t|} = 1\end{aligned}$$

since $t \geq 0$.

Step 4: Next we want to show that for all $x \in X$

$$\lim_{n\to\infty} \mathbf{T}_n(t)x = \mathbf{T}(t)x \tag{6.8}$$

exists and the limit is uniform on bounded sets of t. From theorem 5.1 (i) we have that $\mathbf{A}_n$ and $\mathbf{A}_m$ commute with $\mathbf{T}_n(t)$ and $\mathbf{T}_m(t)$ for any t. So we can take the derivative as defined in (5.6) and get

$$\frac{d}{dt}\mathbf{T}_n(s-t)\mathbf{T}_m(t)x = \mathbf{T}_n(s-t)\mathbf{T}_m(t)[\mathbf{A}_m - \mathbf{A}_n]x$$

using the fact that $\mathbf{T}_n(t)$ is a contraction as shown in Step 3 yields

$$\begin{aligned}
&\int_0^s \frac{d}{dt}\mathbf{T}_n(s-t)\mathbf{T}_m(t)x ds = \int_0^s \mathbf{T}_n(s-t)\mathbf{T}_m(t)[\mathbf{A}_m - \mathbf{A}_n]x ds\\
\Longrightarrow\ & [\mathbf{T}_m(s) - \mathbf{T}_n(s)]x = \int_0^s \mathbf{T}_n(s-t)\mathbf{T}_m(t)[\mathbf{A}_m - \mathbf{A}_n]x ds\\
\Longrightarrow\ & \big\|[\mathbf{T}_m(s) - \mathbf{T}_n(s)]x\big\| \leq \int_0^s \big\|\mathbf{T}_n(s-t)\mathbf{T}_m(t)\big\|\big\|[\mathbf{A}_m - \mathbf{A}_n]x\big\| ds\\
\Longrightarrow\ & \big\|[\mathbf{T}_m(s) - \mathbf{T}_n(s)]x\big\| \leq \big\|[\mathbf{A}_m - \mathbf{A}_n]x\big\| \int_0^s ds = s\big\|[\mathbf{A}_m - \mathbf{A}_n]x\big\|\\
\Longrightarrow\ & \big\|[\mathbf{T}_m(s) - \mathbf{T}_n(s)]x\big\| \leq s\big\|[\mathbf{A}_m - \mathbf{A}_n]x\big\|
\end{aligned}$$

and we have from (6.7) that $\mathbf{A}_n x$ is Cauchy so if $s \in [0, S]$ where $S < \infty$ then

$$\big\|[\mathbf{T}_m(s) - \mathbf{T}_n(s)]x\big\| \leq S\big\|[\mathbf{A}_m - \mathbf{A}_n]x\big\|$$

which implies that $\mathbf{T}_n x$ is Cauchy and not dependent on s, so we have (6.8) for all $x \in D(\mathbf{A})$. It follows using an argument similar to the one in Step 1 that the denseness of $D(\mathbf{A})$ implies that we in fact have (6.8) for all $x \in X$.

Step 5: Now we need to show that $\mathbf{T}(t)$ from (6.8) is in fact a semigroup, that we have strong continuity for $\mathbf{T}(t)$, and that $\mathbf{T}(t)$ is a contraction. Since the convergence of $\mathbf{T}_n(t) \to \mathbf{T}(t)$ does not depend on t when t is in some bounded interval; $\mathbf{T}(t)$ is a semigroup since each $\mathbf{T}_n(t)$ is a semigroup. The strong continuity of $\mathbf{T}(t)$ follows since we have strong continuity for each $\mathbf{T}_n(t)$ and we have uniform convergence in t so we can interchange limits, i.e.

$$\begin{aligned}
&\lim_{t\downarrow 0} \mathbf{T}_n(t)x = x \quad \text{strong continuity}\\
\Longrightarrow\ & \lim_{n\to\infty}\left[\lim_{t\downarrow 0} \mathbf{T}_n(t)x\right] = x\\
\Longrightarrow\ & \lim_{t\downarrow 0}\left[\lim_{n\to\infty} \mathbf{T}_n(t)x\right] = x \quad \text{by uniform convergence in } t\\
\Longrightarrow\ & \lim_{t\downarrow 0} \mathbf{T}(t)x = x
\end{aligned}$$

so we have strong continuity of $\mathbf{T}(t)$. For the contraction using Step 3 have

$$\begin{aligned}
&\|\mathbf{T}_n(t)\| \leq 1\\
\Longrightarrow\ & \|\mathbf{T}_n(t)x\| \leq \|\mathbf{T}_n(t)\|\|x\| \leq \|x\|\\
\Longrightarrow\ & \|\lim_{n\to\infty} \mathbf{T}_n(t)x\| \leq \|x\| \quad \text{by continuity of the norm}\\
\Longrightarrow\ & \|\mathbf{T}(t)x\| \leq \|x\|
\end{aligned}$$

so we have that $\mathbf{T}(t)$ is a contraction.

Step 6: Finally we must show that $\mathbf{A}$ is the infinitesimal generator of $\mathbf{T}(t)$. Apply the integral (5.9) to $\mathbf{T}_n(t)$:

$$\mathbf{T}_n(t)x - x = \int_0^t \mathbf{T}_n(s)\mathbf{A}_n x ds$$

Let $x \in D(\mathbf{A})$ then

$$\lim_{n\to\infty} \mathbf{T}_n(t)x - x = \lim_{n\to\infty} \int_0^t \mathbf{T}_n(s)\mathbf{A}_n x ds$$
$$\Longrightarrow \mathbf{T}(t)x - x = \int_0^t \mathbf{T}(s)\mathbf{A}x ds$$

by (6.7) and (6.8). Then denote by $\mathbf{G}$ the infinitesimal generator of $\mathbf{T}(t)$. Then

$$\lim_{t\downarrow 0} \frac{\mathbf{T}(t)x - x}{t} = \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \mathbf{T}(s)\mathbf{A}x ds$$
$$\Longrightarrow \mathbf{G}x = \mathbf{T}(0)\mathbf{A}x = \mathbf{A}x$$

So for all $x \in D(\mathbf{A})$

$$\mathbf{G}x = \mathbf{A}x$$

which implies that $D(\mathbf{A}) \subseteq D(\mathbf{G})$ and $\mathbf{A} = \mathbf{G}$ when x is restricted to $D(\mathbf{A})$. So G is an *extension* of $\mathbf{A}$ to $D(\mathbf{G})$. But by assumption $\rho(\mathbf{A})$ contains the positive reals and by theorem 5.1 (v) so does $\rho(\mathbf{G})$. By analytic continuation we have that $D(\mathbf{A}) = D(\mathbf{G})$.

$\square$

APPENDIX A. CAUCHY PRODUCT

The Cauchy product is a discrete convolution and is defined as

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k}$$

if the first two sums are absolutely convergent then the resulting product is also absolutely convergent.

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