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Estimation of errors in partial Mueller matrix polarimeter calibration

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ABSTRACT

While active polarimeters have been shown to be successful at improving discriminability of the targets of interest from their background in a wide range of applications, their use can be problematic for cases with strong bandwidth constraints. In order to limit the number of performed measurements, a number of successive studies have developed the concept of partial Mueller matrix polarimeters (pMMPs) into a competitive solution. Like all systems, pMMPs need to be calibrated in order to yield accurate results. In this treatment we provide a method by which to select a limited number of reference objects to calibrate a given pMMP design. To demonstrate the efficacy of the approach, we apply the method to a sample system and show that, depending on the kind of errors present within the system, a significantly reduced number of reference objects measurements will suffice for accurate characterization of the errors.

Keywords: Calibration, Polarimetry, Polarization, Partial Mueller Matrix, Optimization

1. INTRODUCTION

Because partial Mueller matrix polarimeters (pMMPs) are a rather new class of polarimeters,[1,2] they have not yet received enough attention in terms of having a rigorous and fully developed set of best-practices for calibration. Virtually all of the previous polarimeter calibration treatments have focused on full polarimeters,[3,4] which often have a very restricted set of measurements to accomplish well-conditioned results. A consequence of that restriction is a presence of symmetries that allows for a simplified treatment of systems, and one example of such simplification can be seen in Compain's treatment.[4] The fact that Compain's treatment forced PSG and PSA to have four unique states each for a total of 16 measurements, resulted in an elegant derivation that achieved calibration with four reference objects. However, adapting Compain's method to pMMPs is overly constraining because the most effective pMMP designs rely on having additional degrees of freedom associated with unique PSG–PSA measurement conditions.

In an earlier publication,[5] we introduced a family of 10 classes of pMMPs as shown in Figure 1a. The pMMPs were based on having three groups of measurements that guaranteed orthogonal additions to the system's sensor space, thereby maintaining the efficient property of the rank of the measurement matrix, R, equaling the number of measurements made, N. After optimizing the newly introduced set of pMMP designs for the scene space derived from data of Hoover and Tyo,[1] a number of systems showed particularly enticing trade-offs. For that task, it was shown that the optimized 422-pMMP reduced the number of measurements from 16 to eight, while leaving the object-class separability essentially unchanged. The reconstructables matrix for that polarimeter can be seen in Figure 1b.

1.1 Simplistic Polarimeter Calibration

First, examine the polarimetric measurement,

$$I_n = \underline{\mathbf{A}}_n^{\mathrm{T}} \underline{\underline{\mathbf{M}}}\, \underline{\mathbf{G}}_n = \underline{\mathbf{D}}_n'^{\mathrm{T}} \underline{\mathbf{M}}', \tag{1}$$

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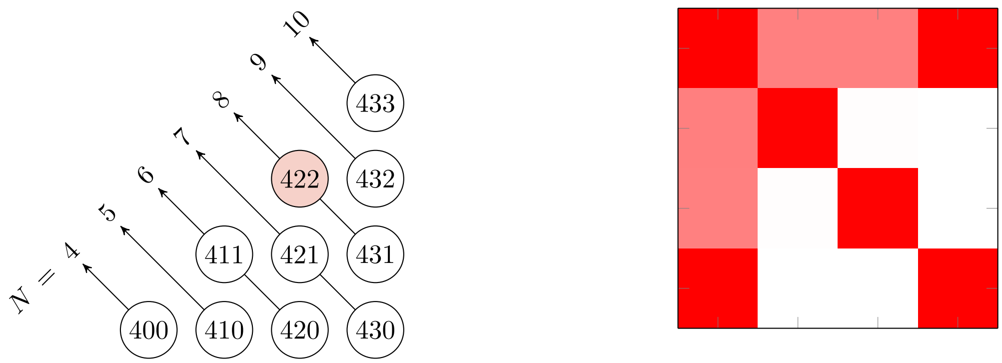

(a) The 10 pMMP classes

(b) Reconstructables matrix for 422

Figure 1: Polarimeters from Alenin and Tyo.[5]

where $\underline{\mathbf{G}}_n$ and $\underline{\mathbf{A}}_n$ are the PSG and PSA Stokes vectors, respectively, and

$$\underline{\mathbf{D}}_n^{'} = \mathrm{vec}(\underline{\underline{\mathbf{D}}}_n) = \mathrm{vec}(\underline{\mathbf{A}}_n^{\mathrm{T}}\underline{\mathbf{G}}_n) = \underline{\mathbf{A}}_n \otimes \underline{\mathbf{G}}_n, \tag{2}$$

$$\underline{\mathbf{M}}^{'} = \mathrm{vec}(\underline{\underline{\mathbf{M}}}), \tag{3}$$

with vec representing a row-by-row vectorization of a matrix. In some of our prior work, we used the primes to denote this matrix-to-vector transformation to encourage a greater amount of attention paid towards the syntax. In this instance, we will omit those primes to avoid every single matrix having one. Thereby, $\underline{\mathbf{D}}^{'} = \underline{\mathbf{D}}$ and $\underline{\mathbf{M}}^{'} = \underline{\mathbf{M}}$. Once we group multiple measurements together, we arrive at the familiar expression,

$$\underline{\mathbf{I}} = \underline{\underline{\mathbf{W}}}\,\underline{\mathbf{M}}. \tag{4}$$

The simplest calibration process can be arrived at by blindly rearranging the expression as

$$\underline{\mathbf{I}}\,\underline{\mathbf{M}}^{-1} = \underline{\underline{\mathbf{W}}}. \tag{5}$$

Obviously, we cannot uniquely invert a Mueller vector, $\underline{\mathbf{M}}$. Instead we need to have multiple objects,

$$\underline{\underline{\mathbf{R}}} = \left[\ \underline{\mathbf{M}}_1 \quad \underline{\mathbf{M}}_2 \quad \cdots \quad \underline{\mathbf{M}}_K\ \right], \tag{6}$$

and rewrite the problem for a set of reference objects,

$$\underline{\underline{\mathbf{I}}}\,\underline{\underline{\mathbf{R}}}^{-1} = \underline{\underline{\mathbf{W}}}. \tag{7}$$

In this instance, $\underline{\underline{\mathbf{W}}}$ can be calculated directly from the known reference objects and the polarimeter intensity measurements. To accomplish this, $\underline{\underline{\mathbf{R}}}$ has to be invertible and well conditioned, which means that we need at least 16 reference objects. This simplistic approach treats $\underline{\underline{\mathbf{W}}}$ as a black box and does not leverage significant knowledge of what the measurements should be based on the system's design. The following sections of this paper will examine our ability to calibrate a polarimeter while taking measurements of the smallest possible number of reference objects, K.

1.2 Polarimeters

In this study, we introduce a flexible method for calibration and use it on two systems in parallel: a) a full 16-measurement Mueller matrix polarimeter, and b) a partial 8-measurement Mueller matrix polarimeter. Both have the same architecture in that their $\underline{\underline{\mathbf{D}}}$ can be described with the following PSG and PSA

$$\underline{\underline{\mathbf{M}}}_{\mathrm{PSG}} = \underline{\underline{\mathbf{M}}}_{\mathrm{LR}}\left(\delta_g, \phi_g\right) \underline{\underline{\mathbf{M}}}_{\mathrm{LD}}(q_g, r_g, \theta_g), \tag{8a}$$

$$\underline{\underline{\mathbf{M}}}_{\mathrm{PSA}} = \underline{\underline{\mathbf{M}}}_{\mathrm{LR}}\left(\delta_a, \phi_a\right) \underline{\underline{\mathbf{M}}}_{\mathrm{LD}}(q_a, r_a, \theta_a), \tag{8b}$$

Polarimeter	Side	q	r	δ	ϕ				θ			
Full	PSG	1.00	0.00	90.00°	96.72°	−14.71°	14.71°	−50.25°	73.49°	−25.74°	25.74°	16.51°
					96.72°	−14.71°	14.71°	−50.25°	73.49°	−25.74°	25.74°	16.51°
					96.72°	−14.71°	14.71°	−50.25°	73.49°	−25.74°	25.74°	16.51°
					96.72°	−14.71°	14.71°	−50.25°	73.49°	−25.74°	25.74°	16.51°
	PSA	1.00	0.00	90.00°	50.25°	50.25°	50.25°	50.25°	73.49°	73.49°	73.49°	73.49°
					−14.71°	−14.71°	−14.71°	−14.71°	64.26°	64.26°	64.26°	64.26°
					36.76°	36.76°	36.76°	36.76°	25.74°	25.74°	25.74°	25.74°
					83.28°	83.28°	83.28°	83.28°	16.51°	16.51°	16.51°	16.51°
422 pMMP	PSG	1.00	0.00	90.00°	32.13°	67.50°	22.52°	−22.52°	−12.86°	−67.50°	−22.48°	22.48°
					47.13°	−42.87°	92.13°	2.13°	47.13°	47.13°	2.13°	2.13°
	PSA	1.00	0.00	90.00°	−26.61°	−26.60°	63.41°	63.41°	108.26°	108.27°	−71.72°	−71.72°
					42.80°	−47.20°	87.79°	−2.02°	42.88°	42.88°	−2.12°	87.88°

Table 1: Idealized parameters for the two polarimeters. The angles, ϕ and θ are shown as a 4×4 and 2×4 matrix for compactness. For any given measurement, each sub-cell of ϕ corresponds to identically positioned sub-cell of θ. Note that these parameters were derived through an optimization process from $\underline{\underline{\mathbf{W}}}$ matrices which were calculated in an architecture-agnostic way. As a result, despite the fact that the optimization reached the needed measurement matrices with acceptable convergence, slight numerical variations still exist.

where $\underline{\underline{\mathbf{M}}}_{\text{LR}}$ and $\underline{\underline{\mathbf{M}}}_{\text{LD}}$ are Mueller matrices for linear retarder and linear diattenuator, respectively. The PSG–PSA measurement conditions combine to produce the net dyad product,

$$\underline{\underline{\mathbf{D}}} = \underline{\underline{\mathbf{M}}}^{\text{T}}_{\text{PSA}} \underline{\underline{\mathbf{M}}}_{\text{PSG}}, \tag{9}$$

which is then vectorized for construction of $\underline{\underline{\mathbf{W}}}$. With the architecture at hand, we can express both polarimeters with a set of PSA and PSA parameters:

$$\vec{\xi}_g = \left\{ \begin{array}{ccccc} q_g & r_g & \delta_g & \vec{\theta}_g & \vec{\phi}_g \end{array} \right\}, \tag{10a}$$

$$\vec{\xi}_a = \left\{ \begin{array}{ccccc} q_a & r_a & \delta_a & \vec{\theta}_a & \vec{\phi}_a \end{array} \right\}, \tag{10b}$$

where q and r are the transmittances for the two orthogonal eigenpolarizations of the polarizer, δ is the retardance, ϕ is the orientation of the retarder and θ is the orientation of the polarizer. When expanded, we have a total of 70 parameters describing our full Mueller matrix polarimeter, and 38 parameters describing our 422 pMMP. The measurement matrices for both can be seen below:

$$\underline{\underline{\mathbf{W}}}^{\text{full}}_{\text{ideal}} = \begin{bmatrix}
0.500 & -0.289 & 0.188 & 0.363 & -0.289 & 0.167 & -0.108 & -0.209 & 0.188 & -0.108 & 0.070 & 0.136 & -0.363 & 0.209 & -0.136 & -0.263 \\
0.500 & 0.289 & -0.363 & 0.188 & -0.289 & -0.167 & 0.209 & -0.108 & 0.188 & 0.108 & -0.136 & 0.070 & -0.363 & -0.209 & 0.263 & -0.136 \\
0.500 & 0.289 & 0.363 & -0.188 & -0.289 & -0.167 & -0.209 & 0.108 & 0.188 & 0.108 & 0.136 & -0.070 & -0.363 & -0.209 & -0.263 & 0.136 \\
0.500 & -0.289 & -0.188 & -0.363 & -0.289 & 0.167 & 0.108 & 0.209 & 0.188 & -0.108 & -0.070 & -0.136 & -0.363 & 0.209 & 0.136 & 0.263 \\
0.500 & -0.289 & 0.188 & 0.363 & 0.289 & -0.167 & 0.108 & 0.209 & -0.363 & 0.209 & -0.136 & -0.263 & -0.188 & 0.108 & -0.070 & -0.136 \\
0.500 & 0.289 & -0.363 & 0.188 & 0.289 & 0.167 & -0.209 & 0.108 & -0.363 & -0.209 & 0.263 & -0.136 & -0.188 & -0.108 & 0.136 & -0.070 \\
0.500 & 0.289 & 0.363 & -0.188 & 0.289 & 0.167 & 0.209 & -0.108 & -0.363 & -0.209 & -0.263 & 0.136 & -0.188 & -0.108 & -0.136 & 0.070 \\
0.500 & -0.289 & -0.188 & -0.363 & 0.289 & -0.167 & -0.108 & -0.209 & -0.363 & 0.209 & 0.136 & 0.263 & -0.188 & 0.108 & 0.070 & 0.136 \\
0.500 & -0.289 & 0.188 & 0.363 & 0.289 & -0.167 & 0.108 & 0.209 & 0.363 & -0.209 & 0.136 & 0.263 & 0.188 & -0.108 & 0.070 & 0.136 \\
0.500 & 0.289 & -0.363 & 0.188 & 0.289 & 0.167 & -0.209 & 0.108 & 0.363 & 0.209 & -0.263 & 0.136 & 0.188 & 0.108 & -0.136 & 0.070 \\
0.500 & 0.289 & 0.363 & -0.188 & 0.289 & 0.167 & 0.209 & -0.108 & 0.363 & 0.209 & 0.263 & -0.136 & 0.188 & 0.108 & 0.136 & -0.070 \\
0.500 & -0.289 & -0.188 & -0.363 & 0.289 & -0.167 & -0.108 & -0.209 & 0.363 & -0.209 & -0.136 & -0.263 & 0.188 & -0.108 & -0.070 & -0.136 \\
0.500 & -0.289 & 0.188 & 0.363 & -0.289 & 0.167 & -0.108 & -0.209 & -0.188 & 0.108 & -0.070 & -0.136 & 0.363 & -0.209 & 0.136 & 0.263 \\
0.500 & 0.289 & -0.363 & 0.188 & -0.289 & -0.167 & 0.209 & -0.108 & -0.188 & -0.108 & 0.136 & -0.070 & 0.363 & 0.209 & -0.263 & 0.136 \\
0.500 & 0.289 & 0.363 & -0.188 & -0.289 & -0.167 & -0.209 & 0.108 & -0.188 & -0.108 & -0.136 & 0.070 & 0.363 & 0.209 & 0.263 & -0.136 \\
0.500 & -0.289 & -0.188 & -0.363 & -0.289 & 0.167 & 0.108 & 0.209 & -0.188 & 0.108 & 0.070 & 0.136 & 0.363 & -0.209 & -0.136 & -0.263
\end{bmatrix} \tag{11a}$$

$$\underline{\underline{\mathbf{W}}}^{422}_{\text{ideal}} = \begin{bmatrix}
0.500 & 0.000 & 0.000 & 0.500 & 0.002 & 0.000 & 0.000 & 0.002 & 0.001 & 0.000 & 0.000 & 0.001 & 0.500 & 0.000 & 0.000 & 0.500 \\
0.500 & -0.000 & -0.000 & -0.500 & 0.002 & -0.000 & -0.000 & -0.002 & 0.001 & -0.000 & -0.000 & -0.001 & 0.500 & -0.000 & -0.000 & -0.500 \\
0.500 & 0.000 & 0.000 & 0.500 & -0.002 & -0.000 & -0.000 & -0.002 & -0.001 & -0.000 & -0.000 & -0.001 & -0.500 & -0.000 & -0.000 & -0.500 \\
0.500 & -0.000 & -0.000 & -0.500 & -0.002 & 0.000 & 0.000 & 0.002 & -0.001 & 0.000 & 0.000 & 0.001 & -0.500 & 0.000 & 0.000 & 0.500 \\
0.500 & -0.037 & 0.499 & -0.000 & 0.037 & -0.003 & 0.037 & -0.000 & 0.499 & -0.037 & 0.498 & -0.000 & -0.001 & 0.000 & -0.001 & 0.000 \\
0.500 & 0.037 & -0.499 & 0.000 & -0.037 & -0.003 & 0.037 & -0.000 & -0.499 & -0.037 & 0.498 & -0.000 & 0.001 & 0.000 & -0.001 & 0.000 \\
0.500 & -0.499 & -0.037 & 0.000 & -0.499 & 0.498 & 0.037 & -0.000 & 0.037 & -0.037 & -0.003 & 0.000 & 0.002 & -0.002 & -0.000 & 0.000 \\
0.500 & 0.499 & 0.037 & -0.000 & 0.499 & 0.498 & 0.037 & -0.000 & -0.037 & -0.037 & -0.003 & 0.000 & -0.002 & -0.002 & -0.000 & 0.000
\end{bmatrix} \tag{11b}$$

Table 1 details the ideal parameters for both polarimeters.

2. TECHNIQUE

The technique outlined in this paper assumes that the design of the polarimeter is known and is not treated as a black box. This is a reasonable assumption to make since the calibration process is often done at instrument's construction. Here, we treat the polarimeter as an idealized set of parameters that define the measurement matrix, $\underline{\underline{\mathbf{W}}}$. Those parameters are then deviated from their ideal values, $\vec{\xi}_{\mathrm{ideal}}$, to their true values, $\vec{\xi}_{\mathrm{true}}$. The calibration is then a process of finding a calibrated set of parameters, $\vec{\xi}_{\mathrm{cal}}$, which approximates $\vec{\xi}_{\mathrm{true}}$. Provided that the system is constructed well, the ideal parameters should be pretty close to true values, allowing us to limit the search space. This is akin to other treatments.[6–8] We can redefine the process in terms of errors (or offsets):

$$\vec{\xi}_{\mathrm{ideal}} \rightarrow \underline{\underline{\mathbf{W}}}_{\mathrm{ideal}} \tag{12a}$$

$$\vec{\xi}_{\mathrm{ideal}} + (\Delta\vec{\xi})_{\mathrm{true}} \rightarrow \underline{\underline{\mathbf{W}}}_{\mathrm{true}} \tag{12b}$$

$$\vec{\xi}_{\mathrm{ideal}} + (\Delta\vec{\xi})_{\mathrm{cal}} \rightarrow \underline{\underline{\mathbf{W}}}_{\mathrm{cal}} \tag{12c}$$

In order to evaluate the proximity of the calibrated matrix to the true matrix, we use the Frobenius norm of the difference between what the calibrated polarimeter reconstructs and what it should reconstruct,

$$\varepsilon = \left\| \underline{\underline{\mathbf{W}}}^{+}_{\mathrm{cal}} \underbrace{\underline{\underline{\mathbf{W}}}_{\mathrm{true}} \underline{\underline{\mathbf{R}}}}_{\underline{\underline{\mathbf{I}}}} - \underline{\underline{\mathbf{W}}}^{+}_{\mathrm{true}} \underbrace{\underline{\underline{\mathbf{W}}}_{\mathrm{true}} \underline{\underline{\mathbf{R}}}}_{\underline{\underline{\mathbf{I}}}} \right\|_{\mathrm{Fro}} . \tag{13}$$

Essentially, Eq. (**??**) adds up $\sum_{\underline{\underline{\mathbf{R}}}} |m_{ij}^{\mathrm{measured}} - m_{ij}^{\mathrm{should\ be}}|^2$ without any preferential weighting. Thought one might be eager to simplify away the $\underline{\underline{\mathbf{W}}}^{+}_{\mathrm{true}}\underline{\underline{\mathbf{W}}}_{\mathrm{true}}$ term, it is necessary to provide pMMP-element-masking that ensures that only the reconstructable Mueller elements of $\underline{\underline{\mathbf{R}}}$ are included within the metric.

3. EXAMPLE

3.1 Reference Objects

To demonstrate this method, we use a reference object comprised of a linear polarizer sandwiched between two retarders,

$$\underline{\underline{\mathbf{M}}}_{\mathrm{ref}} = \underline{\underline{\mathbf{M}}}_{\mathrm{LR}}(\delta_2, \phi_2) \underline{\underline{\mathbf{M}}}_{\mathrm{LP}}(\theta) \underline{\underline{\mathbf{M}}}_{\mathrm{LR}}(\delta_1, \phi_1), \tag{14}$$

the vectorized version of which is

$$\underline{\underline{\mathbf{M}}}_{\mathrm{ref}} = \frac{1}{2} \begin{bmatrix} 1 \\ c(2\theta)(c(2\phi_1)^2 + c(\delta_1)s(2\phi_1)^2) - c(2\phi_1)s(2\phi_1)s(2\theta)(c(\delta_1) - 1) \\ s(2\theta)(c(\delta_1)c(2\phi_1)^2 + s(2\phi_1)^2) - c(2\phi_1)c(2\theta)s(2\phi_1)(c(\delta_1) - 1) \\ c(2\phi_1)s(2\theta)s(\delta_1) - c(2\theta)s(2\phi_1)s(\delta_1) \\ c(2\theta)(c(2\phi_2)^2 + c(\delta_2)s(2\phi_2)^2) - c(2\phi_2)s(2\phi_2)s(2\theta)(c(\delta_2) - 1) \\ (c(2\theta)^2(c(2\phi_1)^2 + c(\delta_1)s(2\phi_1)^2) - c(2\phi_1)c(2\theta)s(2\phi_1)s(2\theta)(c(\delta_1) - 1))(c(2\phi_2)^2 + c(\delta_2)s(2\phi_2)^2) - c(2\phi_2)s(2\phi_2)(c(\delta_2) - 1)(c(2\theta)s(2\theta)(c(2\phi_1)^2 + c(\delta_1)s(2\phi_1)^2) - c(2\phi_1)s(2\phi_1)s(2\theta)^2(c(\delta_1) - 1)) \\ (c(2\theta)s(2\theta)(c(\delta_1)c(2\phi_1)^2 + s(2\phi_1)^2) - c(2\phi_1)c(2\theta)^2s(2\phi_1)(c(\delta_1) - 1))(c(2\phi_2)^2 + c(\delta_2)s(2\phi_2)^2) - c(2\phi_2)s(2\phi_2)(c(\delta_2) - 1)(s(2\theta)^2(c(\delta_1)c(2\phi_1)^2 + s(2\phi_1)^2) - c(2\phi_1)c(2\theta)s(2\phi_1)s(2\theta)(c(\delta_1) - 1)) \\ -(c(2\theta)^2s(2\phi_1)s(\delta_1) - c(2\phi_1)c(2\theta)s(2\theta)s(\delta_1))(c(2\phi_2)^2 + c(\delta_2)s(2\phi_2)^2) - c(2\phi_2)s(2\phi_2)(c(\delta_2) - 1)(c(2\phi_1)s(2\theta)^2s(\delta_1) - c(2\theta)s(2\phi_1)s(2\theta)s(\delta_1)) \\ s(2\theta)(c(\delta_2)c(2\phi_2)^2 + s(2\phi_2)^2) - c(2\phi_2)c(2\theta)s(2\phi_2)(c(\delta_2) - 1) \\ (c(2\theta)s(2\theta)(c(2\phi_1)^2 + c(\delta_1)s(2\phi_1)^2) - c(2\phi_1)s(2\phi_1)s(2\theta)^2(c(\delta_1) - 1))(c(\delta_2)c(2\phi_2)^2 + s(2\phi_2)^2) - c(2\phi_2)s(2\phi_2)(c(\delta_2) - 1)(c(2\theta)^2(c(2\phi_1)^2 + c(\delta_1)s(2\phi_1)^2) - c(2\phi_1)c(2\theta)s(2\phi_1)s(2\theta)(c(\delta_1) - 1)) \\ (s(2\theta)^2(c(\delta_1)c(2\phi_1)^2 + s(2\phi_1)^2) - c(2\phi_1)c(2\theta)s(2\phi_1)s(2\theta)(c(\delta_1) - 1))(c(\delta_2)c(2\phi_2)^2 + s(2\phi_2)^2) - c(2\phi_2)s(2\phi_2)(c(\delta_2) - 1)(c(2\theta)s(2\theta)(c(\delta_1)c(2\phi_1)^2 + s(2\phi_1)^2) - c(2\phi_1)c(2\theta)^2s(2\phi_1)(c(\delta_1) - 1)) \\ (c(2\phi_1)s(2\theta)^2s(\delta_1) - c(2\theta)s(2\phi_1)s(2\theta)s(\delta_1))(c(\delta_2)c(2\phi_2)^2 + s(2\phi_2)^2) + c(2\phi_2)s(2\phi_2)(c(\delta_2) - 1)(c(2\theta)^2s(2\phi_1)s(\delta_1) - c(2\phi_1)c(2\theta)s(2\theta)s(\delta_1)) \\ c(2\theta)s(2\phi_2)s(\delta_2) - c(2\phi_2)s(2\theta)s(\delta_2) \\ -c(2\phi_2)s(\delta_2)(c(2\theta)s(2\theta)(c(2\phi_1)^2 + c(\delta_1)s(2\phi_1)^2) - c(2\phi_1)s(2\phi_1)s(2\theta)^2(c(\delta_1) - 1)) + s(2\phi_2)s(\delta_2)(c(2\theta)^2(c(2\phi_1)^2 + c(\delta_1)s(2\phi_1)^2) - c(2\phi_1)c(2\theta)s(2\phi_1)s(2\theta)(c(\delta_1) - 1)) \\ -c(2\phi_2)s(\delta_2)(s(2\theta)^2(c(\delta_1)c(2\phi_1)^2 + s(2\phi_1)^2) - c(2\phi_1)c(2\theta)s(2\phi_1)s(2\theta)(c(\delta_1) - 1)) + s(2\phi_2)s(\delta_2)(c(2\theta)s(2\theta)(c(\delta_1)c(2\phi_1)^2 + s(2\phi_1)^2) - c(2\phi_1)c(2\theta)^2s(2\phi_1)(c(\delta_1) - 1)) \\ -c(2\phi_2)s(\delta_2)(c(2\phi_1)s(2\theta)^2s(\delta_1) - c(2\theta)s(2\phi_1)s(2\theta)s(\delta_1)) - s(2\phi_2)s(\delta_2)(c(2\theta)^2s(2\phi_1)s(\delta_1) - c(2\phi_1)c(2\theta)s(2\theta)s(\delta_1)) \end{bmatrix}, \tag{15}$$

where $\mathrm{c}(x) = \cos(x)$ and $\mathrm{s}(x) = \sin(x)$. Each non-m_{00} Mueller element of that vector has sinusoidal variations, meaning that given appropriate selections for the various parameters, every polarization-element can be "scanned". For these reference objects, the optimization has access to a limited surface of the Mueller matrix space—non-depolarizing diattenuators. For this exercise, we pre-calculated a CN = 3 matrix of 16 reference

Error	Calibration A	Calibration B	Calibration C
Δq	$[-0.1, \quad 0.0]$	$[-0.1, \quad 0.0]$	$[-0.1, \quad 0.0]$
Δr	$[\ \ 0.0, \quad 0.1]$	$[\ \ 0.0, \quad 0.1]$	$[\ \ 0.0, \quad 0.1]$
$\Delta\delta$	$[-18^\circ, \quad 18^\circ]$	$[-18^\circ, \quad 18^\circ]$	$[-18^\circ, \quad 18^\circ]$
$\Delta\theta$	$[-18^\circ, \quad 18^\circ]$	$[-18^\circ, \quad 18^\circ]$	–
$\Delta\phi$	$[-18^\circ, \quad 18^\circ]$	$[-18^\circ, \quad 18^\circ]$	–
Δf_θ	–	$[-0.1, \quad 0.1]$	–
Δf_ϕ	–	$[-0.1, \quad 0.1]$	–
$\Delta\theta_i$	–	–	$N \times [-18^\circ, \quad 18^\circ]$
$\Delta\phi_i$	–	–	$N \times [-18^\circ, \quad 18^\circ]$
Effect on θ	$\theta_i^{\mathrm{cal}} = \theta_i^{\mathrm{ideal}} + \Delta\theta$	$\theta_i^{\mathrm{cal}} = (1 + \Delta f_\theta)\theta_i^{\mathrm{ideal}} + \Delta\theta$	$\theta_i^{\mathrm{cal}} = \theta_i^{\mathrm{ideal}} + \Delta\theta_i$
Effect on ϕ	$\phi_i^{\mathrm{cal}} = \phi_i^{\mathrm{ideal}} + \Delta\phi$	$\phi_i^{\mathrm{cal}} = (1 + \Delta f_\phi)\phi_i^{\mathrm{ideal}} + \Delta\phi$	$\phi_i^{\mathrm{cal}} = \phi_i^{\mathrm{ideal}} + \Delta\phi_i$
Total Number	10	14	$6 + 4N$

Table 2: Verbose description of the error sets considered in this study. Error set A has a single error for each of the five parameters on PSG and PSA sides, with each orientation angle (θ and ϕ) assumed to have the same offset for each measurement. Error set B has the same offsets, but adds an additional spin-rate error. Error set C presents the most difficult scenario where for each of the N measurements the orientation angles are assumed to have an independent error.

objects,

$$\underline{\underline{\mathbf{R}}} = \begin{bmatrix} 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 & 0.500 \\ -0.289 & 0.289 & 0.289 & -0.289 & -0.289 & -0.289 & -0.289 & -0.289 & 0.289 & 0.289 & 0.289 & 0.289 & 0.289 & 0.289 & -0.289 & -0.289 \\ -0.188 & -0.363 & -0.363 & -0.188 & 0.188 & 0.188 & 0.188 & 0.188 & 0.363 & 0.363 & -0.363 & 0.363 & 0.363 & -0.363 & -0.188 & -0.188 \\ 0.363 & -0.188 & -0.188 & 0.363 & -0.363 & -0.363 & -0.363 & -0.363 & 0.188 & 0.188 & -0.188 & 0.188 & 0.188 & -0.188 & 0.363 & 0.363 \\ -0.289 & 0.289 & -0.289 & 0.289 & -0.289 & -0.289 & 0.289 & 0.289 & -0.289 & 0.289 & 0.289 & -0.289 & 0.289 & -0.289 & 0.289 & -0.289 \\ 0.167 & 0.167 & -0.167 & -0.167 & 0.167 & 0.167 & -0.167 & -0.167 & -0.167 & 0.167 & 0.167 & -0.167 & 0.167 & -0.167 & -0.167 & 0.167 \\ 0.108 & -0.209 & 0.209 & -0.108 & -0.108 & -0.108 & 0.108 & 0.108 & -0.209 & 0.209 & -0.209 & -0.209 & 0.209 & 0.209 & -0.108 & 0.108 \\ -0.209 & -0.108 & 0.108 & 0.209 & 0.209 & 0.209 & -0.209 & -0.209 & -0.108 & 0.108 & -0.108 & -0.108 & 0.108 & 0.108 & 0.209 & -0.209 \\ -0.188 & -0.363 & -0.188 & 0.363 & 0.188 & -0.188 & -0.363 & 0.363 & -0.188 & 0.363 & 0.363 & 0.188 & -0.363 & 0.188 & -0.363 & 0.188 \\ 0.108 & -0.209 & -0.108 & -0.209 & -0.108 & 0.108 & 0.209 & -0.209 & -0.108 & 0.209 & 0.209 & 0.108 & -0.209 & 0.108 & 0.209 & -0.108 \\ 0.070 & 0.263 & 0.136 & -0.136 & 0.070 & -0.070 & -0.136 & 0.136 & -0.136 & 0.263 & -0.263 & 0.136 & -0.263 & -0.136 & 0.136 & -0.070 \\ -0.136 & 0.136 & 0.070 & 0.263 & -0.136 & 0.136 & 0.263 & -0.263 & -0.070 & 0.136 & -0.136 & 0.070 & -0.136 & -0.070 & -0.263 & 0.136 \\ -0.363 & 0.188 & -0.363 & -0.188 & 0.363 & -0.363 & 0.188 & -0.188 & -0.363 & -0.188 & -0.188 & 0.363 & 0.188 & 0.363 & 0.188 & 0.363 \\ 0.209 & 0.108 & -0.209 & 0.108 & -0.209 & 0.209 & -0.108 & 0.108 & -0.209 & -0.108 & -0.108 & 0.209 & 0.108 & 0.209 & -0.108 & -0.209 \\ 0.136 & -0.136 & 0.263 & 0.070 & 0.136 & -0.136 & 0.070 & -0.070 & -0.263 & -0.136 & 0.136 & 0.263 & 0.136 & -0.263 & -0.070 & -0.136 \\ -0.263 & -0.070 & 0.136 & -0.136 & -0.263 & 0.263 & -0.136 & 0.136 & -0.136 & -0.070 & 0.070 & 0.136 & 0.070 & -0.136 & 0.136 & 0.263 \end{bmatrix}, \tag{16}$$

the dyad products of which can be visualized as every combination of tetrahedron-in-$\underline{\mathbf{G}}$ and tetrahedron-in-$\underline{\mathbf{A}}$ matched together. This construction guaranteed that by randomly drawing K objects, the condition number of the cropped $\underline{\underline{\mathbf{R}}}$ would be sufficiently low, while the randomness was not significant enough when a large number of instantiations was considered. Another reason for this selection of the reference object is the fact that there exist high-precision options for both linear retarders (Fresnel rhombs) and linear polarizers (Wollaston prisms).

3.2 Error sets

Looking at Eqs. (10a) and (10b), there are five kinds of parameters within PSG and PSA (q, r, δ, θ, ϕ), with each one carrying a possible error. In this study we looked at three different ways of how those errors might apply. We refer to them as Calibration A, B, and C arranged in the order of increasing complexity. Table 2 summarizes those error structures and the prescribes the search range that was used within the optimization.

In addition to errors within the polarimeter, we also consider errors within reference objects themselves. To do that, we further distinguish between $\underline{\underline{\mathbf{R}}}_{\mathrm{ideal}}$ and $\underline{\underline{\mathbf{R}}}_{\mathrm{true}}$. For the generation of $\underline{\underline{\mathbf{R}}}_{\mathrm{true}}$, we add Gaussian noise of a given standard deviation to only the orientations of the linear polarizer and the linear retarder. Thus, we rewrite the metric as,

$$\varepsilon = \left|\left| \underline{\underline{\mathbf{W}}}_{\mathrm{cal}}^{+} \underline{\underline{\mathbf{W}}}_{\mathrm{true}} \underline{\underline{\mathbf{R}}}_{\mathrm{ideal}} - \underline{\underline{\mathbf{W}}}_{\mathrm{true}}^{+} \underline{\underline{\mathbf{W}}}_{\mathrm{true}} \underline{\underline{\mathbf{R}}}_{\mathrm{true}} \right|\right|_{\mathrm{Fro}} . \tag{17}$$

In this optimization, there are two sources of randomness: a) the selected reference object set, and b) the precision to which the orientations within that reference object are known. Performing a large number of instantiations and averaging the results will reveal the underlying trend.

4. RESULTS

To demonstrate the approach, we ran optimizations for both the full polarimeter of Eq. (11a) and the pMMP of Eq. (11b) under error sets A, B and C detailed in Table 2 with $1 \leq K \leq 16$ reference objects and 50 instantiations. Those results were averaged across those instantiations and are reported in Figs. 2–4. The results show that in all cases, fewer-than-16 reference objects are required to calibrate the system. The full polarimeter can be calibrated with seven, three and two, while the pMMP can be calibrated with three, one and one reference objects for error sets A, B and C, respectively.

5. CONCLUSION

In this paper, we showed that we can enumerate and estimate errors within the full Mueller matrix polarimeters, as well as pMMPs. In each calibration scenario it was shown that fewer-than-16 reference objects are required. One particularly interesting result was for pMMP calibrations under scenarios A and B, where one could say that as few as one or two reference object measurements may be enough. This is due to the fact that pMMPs have a smaller sensor space, and are thus more constrained - requiring fewer reference objects as a result.

It is important to stress that this work only makes claims as to the lower bound of the polarimetric component of the laborious calibration process of a system-at-large. No claims are made in regard to photometric or spectral components of calibration. Those will considerations will have to be added on top of the limits presented here.

This work has a number of possible future continuations, such as exploring different error sources, adding intrinsic treatment of wavelength dependence, extending reference object library to beyond diattenuators, and most intriguingly, investigating how this analysis can be applied to channeled systems under the $\underline{\underline{\mathbf{Q}}}$ formalism.[9]

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REFERENCES

[1] B. G. Hoover and J. S. Tyo, "Polarization components analysis for invariant discrimination," *Appl. Opt.* **46**, pp. 8364–8373, Dec 2007.

[2] J. S. Tyo, Z. Wang, S. J. Johnson, and B. G. Hoover, "Design and optimization of partial mueller matrix polarimeters," *Appl. Opt.* **49**, pp. 2326–2333, Apr 2010.

[3] D. H. Goldstein and R. A. Chipman, "Error analysis of a mueller matrix polarimeter," *J. Opt. Soc. Am. A* **7**, pp. 693–700, Apr 1990.

[4] E. Compain, S. Poirier, and B. Drevillon, "General and self-consistent method for the calibration of polarization modulators, polarimeters, and mueller-matrix ellipsometers," *Appl. Opt.* **38**, pp. 3490–3502, Jun 1999.

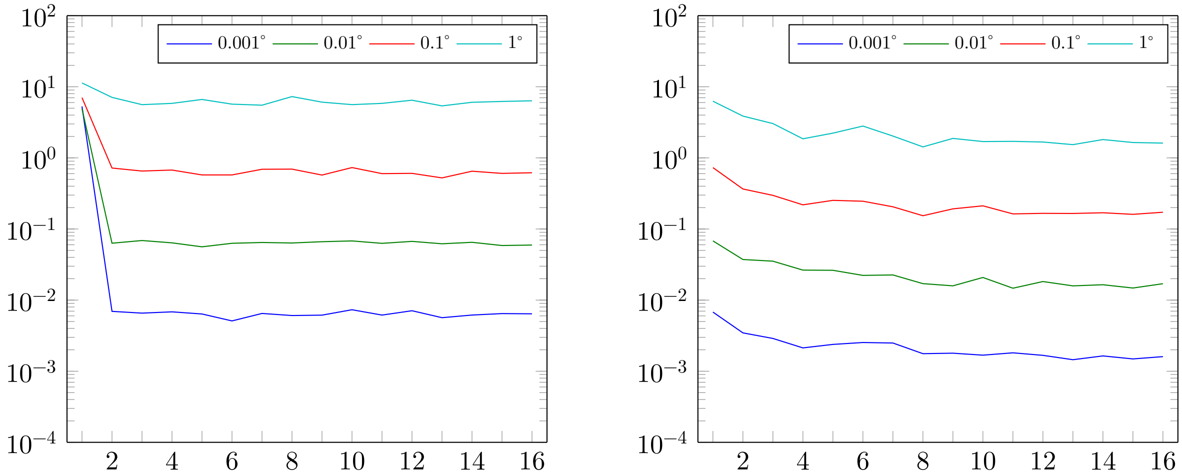
[5] A. S. Alenin and J. S. Tyo, "Structured decomposition design of partial mueller matrix polarimeters," *J. Opt. Soc. Am. A* **32**, pp. 1302–1312, Jul 2015.

[6] C. M. Persons, M. W. Jones, C. A. Farlow, L. D. Morell, M. G. Gulley, and K. D. Spradley, "A proposed standard method for polarimetric calibration and calibration verification," *Proc. SPIE* **6682**, pp. 66820K–66820K–12, 2007.

[7] B. Boulbry, J. C. Ramella-Roman, and T. A. Germer, "Improved method for calibrating a stokes polarimeter," *Appl. Opt.* **46**, pp. 8533–8541, Dec 2007.

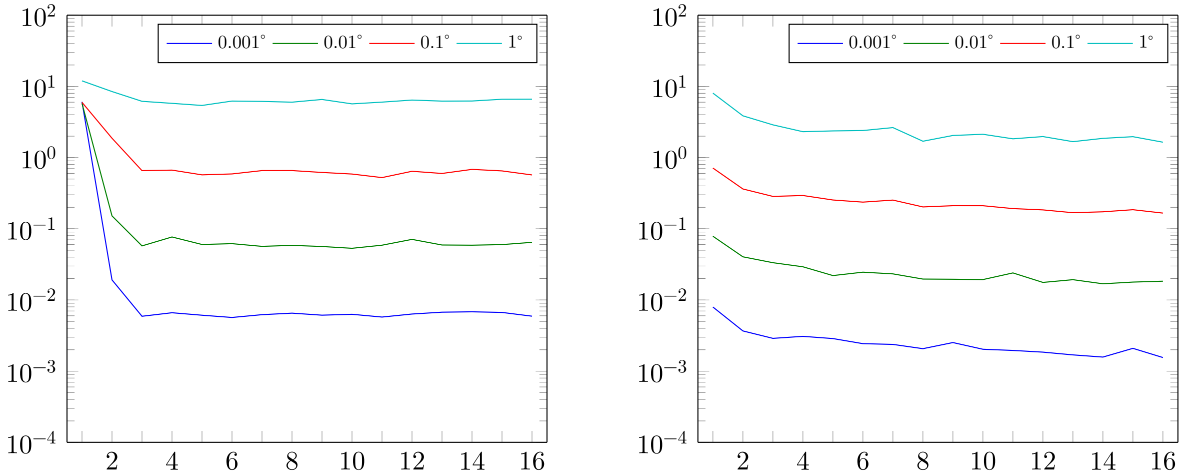
[8] S. B. Powell and V. Gruev, "Calibration methods for division-of-focal-plane polarimeters," *Opt. Express* **21**, pp. 21039–21055, Sep 2013.

[9] A. S. Alenin and J. S. Tyo, "Generalized channeled polarimetry," *J. Opt. Soc. Am. A* **31**, pp. 1013–1022, May 2014.



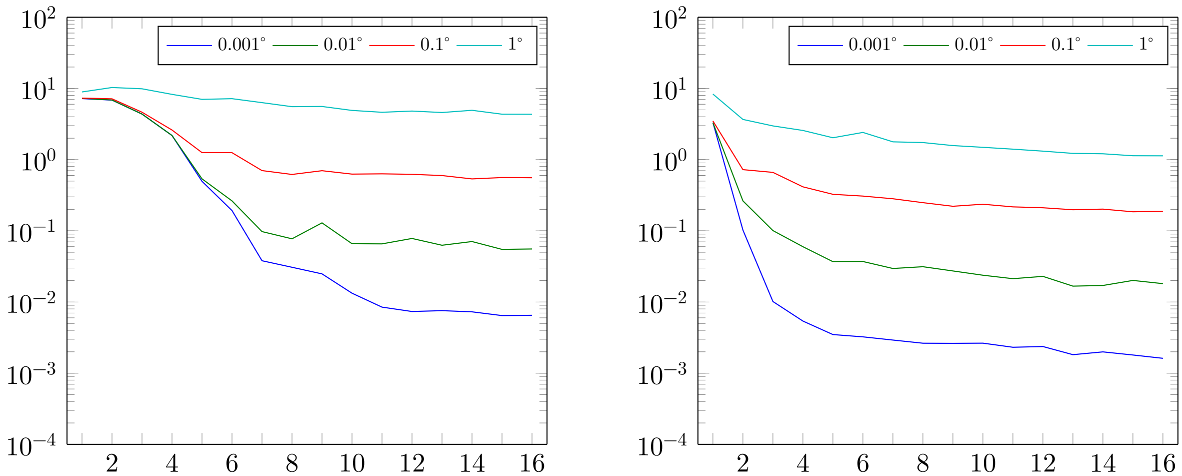
(a) Full (b) pMMP

Figure 2: Calibration A



(a) Full (b) pMMP

Figure 3: Calibration B



(a) Full (b) pMMP

Figure 4: Calibration C